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A CHARACTERISTIC PROPERTY OF THE EXPONENTIAL DISTRIBUTION

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0. SUMMARY

A characterization of the exponential distribution is shown by considering the identical distribution of the random variables, X and $(n-i+1)\ (X_{i,n} - X_{i-1,n})$, where $X_{i,n}$ is the i^{th} order statistics in a random sample of size n.

1. INTRODUCTION

Let X be a random variable (r.v) whose probability density function (p.d.f.) is

$$f\ (x) = \theta^{-1} \exp\ (-x/\theta),\ x > 0,\ \theta > 0 \qquad (1.1)$$

$$= 0,\ \text{otherwise}$$

Suppose $X_1, X_2, \ldots, X_n$ be a random sample of size n from a population with p.d.f. $f(x)$, $x > 0$, and let $X_{1,n} < X_{2,n} < \ldots < X_{n,n}$ be the associated order statistics.

Kotz (1974) discussed extensively the characterizations of exponential distribution by order statistics. Desu (1971) showed that the exponential distribution is the only non-degenerate one with the property that for all K, K times the minimum of the random sample of size K from the distribution has the same distribution as a single observation from the distribution.

Arnold (1971) proved that the characterization is preserved if in Desu's theorema 'for all K' is replaced by two intergers K_1 and K_2 relatively prime and distinct from 1. Puri and Rubin (1970) proved tnat if X_1 and X_2 are independent copies of a r.v. X with p.d.f. $f(x)$, then X and $| X_1 - X_2 |$ nave tne same distribution if and only if f (x) is as given in (1.1).
If $F(x)$ has the density $f(x)$, the ration $h(x)=f(x)/(1-F(x)$ is defined for $F(x) < 1$, is called the hazard rate.
Most materials, structures and devices where replacement is considered wear out with time, the distribution with increasing hazard rate are of interest.
The distribution with decreasing hazard rate may also arise in realiability as reflection of 'work hardering' etc. Barlow, Marsnall and Praschan (1963) considered properties of probability distribution with monotone hazard rate.
In tnis paper we consider a characterization of the exponential distribution bringing the concept of monotone hazard rate.

2. CHARACTERIZATION

Let X be a non-negative r.v. having an absolutely continuous (with respect to Lebesgue measure) strictly increasing distribution function $F(x)$ for all $x \geq 0$, and $F(x) < 1$, for all x. Then the following properties are equivalent:

(a) X has an exponential distribution with density as given in (1.1)

(b) X has a monotone hazard rate and for any fixed i, $2 \leq i \leq n$ and $n \geq 2$, the statistics $(n-i+1)\ (X_{i,n} - X_{i-1,n})$ and X are identically distributed.

<u>Proof</u>: (a) $\Longrightarrow$ (b)

$h(x) = f(x)/(1-F(x)) = \theta^{-1}$. Considering the joint probability of $X_{i,n}$ and $X_{i-1,n}$, and using the transformation $U = X_{i,n}$ and $V = (n-i+1)(X_{i,n} - X_{i-1,n})$, it can be shown that V is identically disbuted as X.

Proof (b) $\Longrightarrow$ (a).

Let $Y_i = X_{i,n} - X_{i-1,n}$, the probability density of Y_i, Pyke (1965), is

$$f_{Y_i}(y) = \frac{n!}{(i-2)!\,(n-i)!} \int_0^\infty (F(u))^{i-2}\ (1-F(u+y))^{n-i}\ f(u)f(u+y)\,du \quad (2.1)$$

Substituing $Z = (n-i+1)\ Y_i$, we get the p.d.f. of Z, as

$$f_Z(Z) = \frac{n!}{(i-2)!(n-i)!} \int_0^\infty (F(u))^{i-2} (1-F(u+ \frac{Z}{n-i+1}))^{n-i} f(u) f(u+ \frac{Z}{n-i+1}) \frac{du}{n-i+1}$$

Since Z and X are identically distributed, we get

$$f(Z) = \frac{n!}{(i-2)!(n-i+1)!} \int_0^\infty (F(u))^{i-2}\ (1-F(u + \frac{Z}{n-i+1}))^{n-i+1} du, \quad (2.3)$$

Writing

$$\frac{(i-2)!(n-i+1)!}{n!} = \int_0^\infty (F(u))^{i-2}\ (1-F(u))^{n-i+1}\ f(u)\,du$$

and rearranging we get from (2.3)

$$0 = \int_0^\infty (F(u))^{i-2} f(u) \{ f(Z)(1-F(u))^{n-i+1} - (1-F(u+\frac{Z}{n-i+1}))^{n-i} f(u+\frac{Z}{n-i+1}) \} du \tag{2.4}$$

for all Z

Integrating (2.4) with respect Z form 0 to Z_1, and interchanging the integration (which is permitted here), we get

$$0 = \int_0^\infty (F(u))^{i-2} (1-F(u))^{n-i+1} G(u, Z_1) du \tag{2.5}$$

for all Z_1, where

$$G(u_1, Z_1) = \left(\frac{1-F(u+\frac{Z_1}{n-i+1})}{1-F(u)}\right)^{n-i+1} - (1-F(Z_1)) \tag{2.6}$$

By Lemma 1, given in Appendix, we have

$G(0, Z_1) = 0$, for all Z_1, i, e,

$$1-F(Z) = (1-F(\frac{Z}{n-i+1}))^{n-i+1} \tag{2.7}$$

for all Z.

Substituing H(Z) = and using $\psi(Z) = -\log H(Z)$,

$$\psi(Z) = (n-i+1)\,\psi\left(\frac{Z}{n-i+1}\right), \text{ for all } Z. \tag{2.8}$$

We know $F(Z)$ is strictly monotone increasing function with $F(Z) < 1$, for all Z, hence $\psi(Z)$ is continuous for all Z, thus the solution of (2.8), Aczel (1966), is $\psi(Z) = CZ$ (2.9)

Where C is a constant, thus $F(Z) = 1 - e^{CZ}$.

Using the boundary condition $F(0) = 0$ and $F(\infty) = 1$, we get

$$F(x) = 1 - e^{-\theta x}$$

where $\theta > 0$. (2.10)

3. APPENDIX

Lemma 1:

Let $F(x)$ be the absolutely continuous (w.r.t. Lebesgue measure) monotone strictly increasing distribution of a non-negative r.v. X, and $F(0) = 0$, $F(x) < 1$, for all x and $f(x)$ the corresponding p.d.f. If the hazard rate $h(x)=f(x)/(1-F(x))$ is monotone and if

$$\int_0^\infty (F(u))^{i-2}(1-F(u))^{n-i+1}\, G(u,Z)\,du = 0, \text{ for all } Z, \tag{A.1}$$

where

$$G(u,Z) = \left(\frac{1-F\left(u+\frac{Z}{n-i+1}\right)}{1-F(u)}\right)^{n-i+1} - (1-F(Z)), \text{ then}$$

$G(0,Z) = 0$, for all Z

<u>Proof</u> we have $G(u,0) = 0 = G(u,\infty)$

$$\frac{\partial G(u,Z)}{\partial u} = (n-i+1)\, q(u,Z)\left(h(u) - h\left(u + \frac{Z}{n-i+1}\right)\right) \tag{A.2}$$

where $q(u,Z) = \left(\dfrac{1-F\left(u+\frac{Z}{n-i+1}\right)}{1-F(u)}\right)^{n-i+1}$

(i) $h(x)$ = constant for all x

$\frac{\partial}{\partial u} G(u,Z) = 0$ for all u and Z, then $G(u,Z)$ is independent of u for all Z, hence by (A.1), $G(u,Z) = 0$ for all u and Z.

(ii) $h(x)$ is strictly decreasing in x.

For strictly monotone decreasing $h(x)$ in x, it follows from (A.2) that $G(u,Z)$ is strictly monotone increasing in u with fixed Z. We know also Barlow and Praschan (1965) that $\log(1-F(u))$ is convex. For (A.1) to be true $G(u,Z)$ for $Z = Z_1$, if $G(u,Z_1) \not\equiv 0$, must be negative for $u_1 \in I$ and is an interval $[o,c]$, where c is a real number. Let $u_1 \in I$, then $G(u_1,Z_1) < 0$.

$$\frac{\partial}{\partial Z_1} G(u_1,Z_1) = f(Z_1) - q(u_1,Z_1)\, h\left(u_1 + \frac{Z_1}{n-i+1}\right)$$

$$= (1-F(Z_1))\left[h(z_1) - h\left(u_1 + \frac{Z_1}{n-i+1}\right) q(u_1,Z_1)/(1-F(Z_1))\right]$$

$$> (1-F(Z_1))\,(h(Z_1) - h(u_1 + \frac{Z_1}{n-i+1}))\,,$$

since $G(u_1, Z_1) < 0$, $\quad q(u_1, Z_1)/(1-F(Z_1)) < 1$,

$$> 0, \text{ if } u_1 + \frac{Z_1}{n-i+1} > Z_1, \text{ i, e, } Z_1 > \frac{n-i+1}{n-i} u_1$$

Thus $G(u_1, Z_1)$, when negative, as a function of Z_1 for fixed u_1 is increasing if $Z_1 > \frac{n-i+1}{n-i} u_1$.

For any arbitrary small positive δ, we may take $u_1 = \frac{n-i}{n-i+1}\delta$ then

for any $Z > \delta$, $G(\frac{n-i}{n-i+1}\delta, Z)$ is in increasing with Z.

But $G(\frac{n-i}{n-i+1}\delta, Z)$ increases to zero for all $Z > \delta$, as $Z \rightarrow \infty$.

$G(\frac{n-i}{n-i+1}\delta, 0) = 0$ for all δ. Hence by continuity of

$G(\frac{n-i}{n-i+1}\delta, Z)$ in Z it follows that $G(\frac{n-i}{n-i+1}\delta, Z) = 0$ for all Z.

Hence $G(0,Z) = 0$ for all Z.

(iii) $h(x)$ is strictly monotone increasing in x. In this case $G(u,Z)$ will be strictly decreasing. Similar to argument in (ii), it can be proved that $G(0, Z) = o$ for all Z.

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