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ON A CHARACTERIZATION OF THE EXPONENTIAL DISTRIBUTION BY ORDER STATISTICS

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0. SUMMARY

A characterization of the exponential distribution is shown by considering the identical distribution of the random variables $n X_{1,n}$ and $(n-i+1)(X_{i,n}-X_{i-1,n})$, where $X_{j,n}$ is the jth order statistic, $(j=1, i, i-1)$ is a sample of size n.

1. INTRODUCTION

Let be a random variable (r.v) whose probability density function (p.d.f.) is

$$f(x) = \theta^{-1} \exp(-x/\theta), \quad x>0 \quad \theta>0$$

$$= 0, \text{ otherwise.} \tag{1.1}$$

Suppose $X_1, X_2, \ldots, X_n$ be a random sample of size n from a population with p.d.f. $f(x)$, $X > 0$, and let $X_{1,n} < X_{2,n} < \ldots < X_{n,n}$ be the associated order statistics. Kotz (1974) and Galambos (1975) discussed extensively the characterization of exponential distribution by order statistics. Desu (1971) showed that the exponential distribution is the only non-degenerate one that for all K, K times the minimum of the random sample of size K from the distribution has the same distribution as a single observation. Sethuraman (1965) proved a stronger form of Desu's theorem. Puri and Rubin (1970) proved that if X_1, X_2 are independent copies of a r.v. X with p.d.f. $f(x)$, then X and $|X_1-X_2|$ have the same distribution if and only if $f(x)$ as given in (1.1). If $F(x)$ has the density $f(x)$, the ratio $h(x)=f(x) / (1-F(x)$, for $F(x) < 1$, is called the hazard rate. Most materials, structures and devices where replacenment is considered wear out with time, the distribution with increasing hazard rate are of interest.

The distribution with decreasing hazard rale may also avise in realiability as reflection of 'work hardening' etc. Barlow, Marshall and Pröschan (1963) considered properties of probability distribution with monotone hazard rate. Ahsanullah (1975) gave a characterization of exponential distribution by considering identicall distribution of $n X_{1,n}$ and $X_{n,n} - X_{n-1,n}$. In this paper a generalisation of the author's (1975) work is given.

2. CHARACTERIZATION

Let X be a non-negative r.v. having an absolutely continuous (with respect to lebesgue measure) strictly increasing distribution function F(x) for all x > 0, and F(x) $\leq$ 1, for all x. Then the following properties are equivalent:

(a) X has an exponential distribution with density as given in (1.1)

(b) X has a monotone hazard rate and for any fixed i, and n, $2 \leq i \leq n$, the statistics $(n-i+1)(X_{i,n} - X_{i-1,n})$ and $n X_{1,n}$ are identically distributed.

Proof: (a) $\Longrightarrow$ (b)

Let $h(x) = f(x)/(1-F(x) = \theta^{-1}$. Considering the joint probability of $X_{i,n}$ and $X_{i-1,n}$ and using the transformation $U = X_{i,n}$ and $V = (n-i+1)(X_{i,n} - X_{i-1,n})$ it can be shown, using the value of $h(x) = \theta^{-1}$, that V is identically distributed as $n X_{1,n}$.

Proof: (b) $\Longrightarrow$ (a)

Let $Y_i = X_{i,n}$, the probability density of Y_i, Pyke (1965), is

$$f_{Y_i}(y) = \frac{n!}{(i-2)!(n-i)!} \int_0^\infty (F(u))^{i-2}(1-F(u+y))^{n-i} f(u) f(u+y)\, du \tag{2.1}$$

Substituting $Z = (n-i+1) Y_i$, we get the p.d.f. of Z, as

$$f_Z(z) = \frac{n!}{(i-2)!(n-i+1)!} \int_0^\infty (F(u))^{i-2}(1-F(u+z/(n-i+1)))^{n-i} f(u) f(u+z/(n-i+1))\, du \tag{2.2}$$

The probability density function of $W = nX_{i,n}$ is

$$f_W(w) = (1-F(w/n))^{n-1} f(w/n)$$

Since Z and W are identically distributed, using the fact

$$\frac{(i-2)!(n-i+1)!}{n!} = \int_0^\infty (F(u))^{i-2} (1-F(u))^{n-i+1} f(u)\, du,$$

We get on simplification from (2.2) and (2.3),

$$0 = \int_0^\infty (F(u))^{i-2} (1-F(u))^{n-i+1} f(u)\, g(u,z)\, du, \text{ for all } z, \tag{2.4}$$

Where

$g(u,z_1) =$

$$(1-F(z/n))^{n-1} f(z/n) - (1-F(u+z/(n-i+1))/(1-F(u)))^{n-i} f(u+z/(n-i+1))/ (1-F(u)$$

Integrating (2.4), we respect to z from 0 to z_1, we obtain after some simplification

$$0 = \int_0^\infty (F(u))^{i-2}(1-F(u))^{n-i+1} f(u)\, G(u,z_1)\, du = 0, \text{ for all } z_1 \tag{2.5}$$

where

$$G(u, z_1) = \left(\frac{1-F\left(u+\frac{z_1}{n-i+1}\right)}{1-F(u)}\right)^{n-i+1} - \left(1-F\left(\frac{z_1}{n}\right)\right)^n \quad (2.6)$$

By lemma 1, given in appendix, we have

$G(0, Z_1) = 0$, for all Z_1, i, e,

$$\left(1-F\left(\frac{z}{n-i+1}\right)\right)^{n-i+1} = \left(1-F\left(\frac{z}{n}\right)\right)^n, \text{ for all } z \quad (2.7)$$

Substituting $H(z) = 1-F(z)$, $\Psi(z) = -\log H(z)$ and

$y = \frac{1}{n} z$, we get

$$\Psi(y) = \frac{n-i+1}{n} \Psi\left(\frac{n}{n-i+1} y\right), \text{ for all } y, \quad (2.8)$$

We know $F(z)$ is sttictly monotone increasing function with $F(z) < 1$, for all z, hence $y(z)$ is continuous for all z, thus the solution of (2.8) Aczel (1966), is

$$\Psi(y) = cy \quad (2.9)$$

Where c is constant, thus $F(y) = 1-e^{cy}$

Using the boudary condition $F(0) = 0$, and $F(\infty) = 1$,

we get

$$F(x) = 1-e^{-\theta x} \quad (2.10)$$

Where $\theta > 0$

3. APPENDIX

Lemma I:

Let F (x) be the absolutely continuous (with respect to lebesgue measure) monotone strictly increasing distribution function on a non-negative r.v.X, and F (0) = 0, F (x) < 1, for all x and f(x) be the corresponding p.d.f. If the hazard rate is monotone and if

$$\int_0^\infty (F(u))^{i-2} (1-F(u))^{n-i+1} f(u) \; G(u,z) = 0, \tag{A.1}$$

for fixed i, and n and all z, where

$$G(u,z) = \left(\frac{1-F(u+\frac{z}{n-i+1})}{1-F(u)}\right)^{n-i+1} - (1-F(\tfrac{z}{n}))^n,$$

G (0,z) = 0, for all z.

<u>Proof</u>:

We have G(u,0) = 0 = G(0, ∞)

$$\frac{\partial}{\partial u} G(u,z) = (n-i+1) q(u,z) \left(h(u) - h(u+\frac{z}{n-i+1})\right) \tag{A.2}$$

where

$$q(u,z) = \left(\frac{1-F(u+\frac{z}{n-i+1})}{1-F(u)}\right)^{n-i+1}$$

(i) h (x) = constant for all x.

$\frac{\partial}{\partial u} G(u,z) = 0$, for all u and z, hence G(u,z) is independent of u for all u and z.

(ii) $h(x)$ is strictly monotone decreasing in x.

It follows from (A.2) that $G(u,z)$ is strictly increasing in u, with fixed z. For (A.1) to be true $G(u,z_1)$ for $z=z_1$, if $G(u,z_1) \not\equiv 0$, must be negative in an interval including the pont zero. Let $u \in I$, where I is an interval $[o,b]$, b is a real number. Then $G(u_1, z_1) <0$, for $u \in I$.

$$\frac{\partial}{\partial z_1} G(u_1, z_1) = (1-F(z_1/n))^n \; \ell(u_1,z_1)$$

where

$$\ell(u_1,z_1) = h(z_1) - m(u_1,z_1) \; h(u_1+z_1/(n-i+1))$$

and

$$m(u_1,z_1) = (1/(1-F(z_1/n)))^n \; ((1-F(u_1+z_1/(n-i+1)))/(1-F(u_1)))^n$$

$$G(u_1,z_1) < 0 \implies m(u_1,z_1) < 1$$

Hence for $u \in I$,

$$\frac{\partial}{\partial z_1} G(u_1, z_1) > (1-F(z_1/n))^n \; (h(z_1) - h(u_1+z_1/(n-i+1)))$$
$$> 0$$

Thus $G(u_1,z_1)$ is increasing with z_1. But $G(u_1,0)=0$ and $G(u_1, \infty)=0$, therefore it follows by the continuity of $G(u_1,z_1)$ that $G(u_1,z_1)=0$, for all z_1. Hence $G(0,z_1)=0$, for all z_1.

(iii) $h(x)$ is strictly monotone increasing in x. In this case $G(u,z)$ will be strictly monotone decreasing in u with fixed z. Following the similiar argument as in (ii) it can be shown that $G(0,z)=0$ for all z.

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