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ON SYSTEMS AND ITS ANALYSIS

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0. INTRODUCTION

In this paper attention will be given mainly on the systems that are linear. A system is called linear if (i) an imput $x_1(t)$ produces an output $y_1(t)$ (ii) an imput X_2 (t) produces an output Y_2 (t), then an imput $C_1X_1 + c_2x_2$ (t) produces an output c_1y_1 (t) + c_2y_2 (t) for all pairs of imputs x_1 (t) and x_2 (t) and all pairs of constants c_1 and c_2 we will for convenience classify the linear system as

a) linear system with continuous variable

b) linear system with discrete variables.

We shall denote the independent variable by t. A system in which the imput and output are all continuous is called a continuous system. Most continuous system can be represented by differential equations. Similasly a system with discrete imput and output will be called discrete system. A discrete system mostly will be represented by difference equations.

1. LINEAR CONTINUOUS SYSTEM

(A) It will be convenient first to consider system with one input and one output.

$\dot{X}(t) = a X(t)$, $X(0) = X_0$, where a is a real constant and $\dot{X}(t) = \frac{d}{dt} X(t)$

The solution of the equation is

$$X(t) = e^{at} X(0) \tag{1.1}$$

If (i) $a > 0$, (1.1) will represent an exponential growth

(ii) $a < 0$, (1.1) will represent an exponential growth decay

(iii) $a = 0$, then $X(t) = X(0)$ which means a complete memory of a past value.

$$\log (X(t_1)/X(t_2)) = a (t_1 - t_2)$$

(B) General linear system with constant coefficients

(i) Diagonal system

$$\dot{X}_1(t) = \lambda_1 X_1(t)$$

$$\dot{X}_2 (t) = \lambda_2 X_2 (t) \qquad (1.2)$$

$$\dot{X}_n (t) = \lambda_n X_n (t)$$

Even though the above model represents an n-th order system by a vector of n elements, the n first order differential equations are independent. Writting in matrix notation.

$$\underline{X} (t) = \begin{bmatrix} X_1 (t) \\ X_2 (t) \\ X_n (t) \end{bmatrix}, \underline{\dot{X}}(t) = \begin{bmatrix} \dot{X}_1 (t) \\ \dot{X}_2 (t) \\ \dot{X}_n (t) \end{bmatrix}, \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Defining the matrix exponential as

$$e^{mt} = \sum_{j=0}^{\infty} \frac{(Mt)^j}{j} = I + \underline{M}t + \frac{1}{2} \underline{M}^2 t^2 + \ldots$$

Substituting $M = \Lambda$, a diagonal matrix, we get

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & 0 & 0 & \dots & 0 \\ 0 & \lambda_2^k & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & \lambda_n^k \end{bmatrix}$$ and hence

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix}$$

with $$\underline{X}\,(0) = \begin{bmatrix} X_1\ (0) \\ X_2\ (0) \\ \cdot \\ \cdot \\ \cdot \\ X_n\ (0) \end{bmatrix}$$

the solution of (1.2) is

$$\underline{X}\ (t) = e^{\Lambda t}\ \underline{X}\ (0) \qquad (1.3)$$

B (ii) Non diagonal system

$$X_1 (t) = a_{11} X_1 (t) + a_{12} X_2(t) + \ldots + a_{1n} X_n(t)$$

$$X_2 (t) = a_{21} X_1 (t) + a_{22} X_2(t) + \ldots + a_{2n} X_n(t) \qquad (1.4)$$

$$X_n (t) = a_{n1} X_1 (t) + a_{n2} X_2(t) + \ldots + a_{nn} X_n(t)$$

In the matrix for

$\underline{X} (t) = \underline{A} \, \underline{X} (t)$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots\cdots & a_{nn} \end{bmatrix}$$

the elements of $\underline{A}$ are not functions of any of the X'S. We shall show two different methods of the solution of the equation (1.4). First by converting the system to a diagonal one and secondly by the use of laplace transformation. Let T be a non singular matrix such that $T'AT = \Lambda$, where is a diagonal matrix. Let $\underline{X}^* = T^{-1} X$, then

$$\frac{d}{dt} X^* (t) = T^{-1} X = T^{-1} A X(t) = T^{-1} AT \underline{X}^* (t) = \Lambda \underline{X}^* (t) \qquad (1.5)$$

Thus (1.5) is a diagonal system on the transformed variables, solution of (1.5) is

$$X^{*}\ (t) = e\Lambda^{t}\ \underline{X}^{*}\ (0)$$

Writting $\underline{T} = (t_{i_j})$, we get

$$\begin{bmatrix} X_1\ (t) \\ \cdot \\ \cdot \\ \cdot \\ X_n\ (t) \end{bmatrix} = T \begin{bmatrix} X_1^{*}\ (t) \\ \cdot \\ \cdot \\ \cdot \\ X_n^{*}\ (t) \end{bmatrix} \begin{bmatrix} t_{11}\ e^{\lambda_1 t}\ X_1^{*}\ (0) + \ldots + t_{1n}\ e^{\lambda n t}\ X_n^{*}\ (0) \\ \\ t_{n1}\ e^{\lambda n t}\ X_1^{*}\ (0) + \ldots + t_{nn}\ e^{\lambda n t}\ X_n^{*}\ (0) \end{bmatrix}$$

$$= \begin{bmatrix} t_{11}\ X_1^{*}\ (0) \\ \cdot \\ \cdot \\ \cdot \\ t_{n1}\ X_n^{*}\ (0) \end{bmatrix} e^{\lambda_1 t} + \ldots + \begin{bmatrix} t_{1n}\ X^{*}\ (0) \\ \cdot \\ \cdot \\ \cdot \\ t_{nn}\ X_n^{*}\ (0) \end{bmatrix} e^{\lambda n t}$$

$$\underline{X}(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_n(t) \end{bmatrix} , \quad \dot{\underline{X}}(t) = \frac{d}{dt}\underline{X}(t) = \begin{bmatrix} \dot{X}_1(t) \\ \dot{X}_2(t) \\ \vdots \\ \dot{X}_n(t) \end{bmatrix}$$

$$\underline{A}(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & & \vdots \\ a_{n1}(t) & a_{n2}(t) & & a_{nn}(t) \end{bmatrix}$$

The other form of linear system with time dependent coefficients may be as follows:

$$X^{(n)}(t) = a_{n-1} X^{(n-1)}(t) + a_{n-2} X^{n-2}(t) + \dots + a_1(t) X^1(t) \qquad (1.10)$$

where

$$X^1(t) = X_1(t),\ X^{(2)}_{(t)} = X_2(t), \dots,\ X^{(n-1)} = X_{n-1}(t)$$

The above will give

$X(t) = 0 + X_1 (t)$

$X_1(t) = 0 + 0 + X_2(t)$

$X_{n-2} (t) = 0 + 0 + 0 + 0 \dots + X_{n-1} (t)$

$X_{n-1}(t) = a_0 (t) X(t) + a_1 (t) X_1(t) +\dots+ a_{n-1}(t)$

or in matrix notation

$$\dot{\underset{\sim}{X}} (t) = A(t) X(t) \tag{1.11}$$

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ . & . & . & . & \dots & . \\ a_0 (t) & a_1 (t) & a_2(t) & a_3(t) & \dots & a_{n-1}(t) \end{pmatrix}$$

Let $X_1(t)$, $X_2(t)$,..., $X_n(t)$ be a set of independent solution of the differential equation of degree n of (1.10) then $X_1(t)$, $X_2(t)$,..., $X_n(t)$ will be

$$X_1(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \\ X_n(t) \end{pmatrix}, X_2(t) = \begin{pmatrix} X_2(t) \\ X_2^2(t) \\ X_n^2(t) \end{pmatrix}, X_n(t) = \begin{pmatrix} X_1^n(t) \\ X_2^n(t) \\ X_n^n(t) \end{pmatrix}$$

Linear system with forced imput

$$\dot{X}_1(t) = a_{11} X_1(t) + a_{12} X_2(t) + \ldots + a_{1n} X_n(t) + \quad (t)$$

$$\dot{X}_2(t) = a_{21} X_1(t) + a_{22} X_2(t) + \ldots + a_{2n} X_n(t) + r_2(t) \qquad (1.12)$$

$$\dot{X}_n(t) = a_{n1} X_1(t) + a_{n2}(t) + \ldots + a_{nn} X_n(t) + r_n(t)$$

a's are independent of t

In matrix form (1.12) can be written as

$$\dot{X}(t) = A\, X(t) + R\ (t) \qquad (1.13)$$

$$X(t) = W(t)\ X\ (0) + \int_0^t W(t-c)\ R(c)\ dc$$

where $W\ (t) = \mathcal{L}^{-1} (SI-A)^{-1}$

Example:

$$\dot{X}_1(t) = X_2(t) + r_1\ (t)$$

$$\dot{X}_2(t) = 2_{X1}(t) - 3\ X_2(t)\ r_2(t)$$

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -5 \end{pmatrix} \qquad R(t) = \begin{pmatrix} r_1(t) \\ r_2(t) \end{pmatrix}$$

$$SI-A = \begin{pmatrix} S & -1 \\ 2 & S+3 \end{pmatrix} , \quad (SI-A)^{-1} = \frac{1}{S^2 + 35 + 2} \begin{pmatrix} S+3 & 1 \\ -2 & S \end{pmatrix}$$

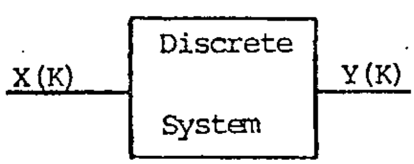
$$W(t) = \mathcal{L}^{-1} (SI-A)^{-1} = \begin{pmatrix} (2e^{-t} - e^{-2t}) & (e^{-t} - e^{-2t}) \\ -2(e^{-t} - e^{-2t}) & (2e^{-2t} - e^{t}) \end{pmatrix}$$

$$W(t-c) = \begin{pmatrix} (2e^{-(t-c)} - e^{-2(t-c)} & (e^{-(t-c)} - e^{-2(t-c)} \\ -2(e^{(t-c)} - e^{-2(t-c)} & (2\, e^{-2(t-c)} - e^{-(t-c)} \end{pmatrix}$$

2. Linear Discrete System

(A) Discrete system with constant coefficients

(i) For convenience we will consider in the begning the system with on input one output.



$Y(K) - Y(k-1) = X\ (k)$, α is independent of k,

$$X(0) = 1$$

$$X(k) = 0 \text{ for } k \neq 0,$$

$$Y(m) = 0 \text{ if } m < 0$$

$Y(0) = 1$

$Y(1) - \alpha\ Y(0) = X(1)$

$Y(1) = \alpha$

$Y(2) = \alpha^2, \ldots\ Y(k) = \alpha^k$

Thus the solution is

$$Y(k) = \alpha^k \text{ for } k > 0$$

$$= 0 \text{ for } k < 0$$

(ii) general discrete system with constant coefficients

$$\begin{bmatrix} X_1(k+1) \\ X_2(k+1) \\ \vdots \\ X_n(k+1) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} X_1(k) \\ X_2(k) \\ \vdots \\ X_n(k) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} r(k)$$

In matrix notation

$$\underline{X}\ (k+1) = \underline{A}\ \underline{X}\ (k) + \underline{B}\ r\ (k)$$

$$X\ (1) = A(0) + B\ r\ (0)$$

$$X\ (2) = A\ X\ (1) + B\ r\ (1) = A^2\ X(0) + AB\ r(0) + Br\ (1)$$

$$X\ (k) = A^k\ X(0) + \sum_{j=0}^{k-1} A^j\ B\ r\ (k-m-1)$$

Linear discrete system with time dependent coefficients:

The discussion of this type of systems and their solutions will not be discussed here but intend to discuss in some other occasion.

Summary and Discussion

The identification of system and subsequent analysis involves a good deal of computer work. Moreover there may occur some objective function whose minimization a maximization may be of interest. This will add to the complexicity of the problem. However with hard work and will planned procedure fruitful result can alwaus be achived.