



A Monte Carlo Study on the Fit of a Cost Function by a Fourier Flexible Functional Form

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Abstract

We use the simulation model of Chalfant and Gallant (1985) to investigate the fit of a Fourier flexible form to a cost function when interest is to estimate elasticities of substitution. The data is subject to errors in variables and the estimation method is seemingly unrelated regressions. We use two approximating Fourier functional forms with 13 (FFF13) and 22 (FFF22) parameters. The biases in point estimation are negligible for FFF13. Biases and large sample distributional approximations to classical t-statistics deteriorate when one moves from FFF13 to FFF22. When estimates are properly centered and scaled a normal approximation holds.

1 Introduction

The classical use of a flexible functional form in econometrics to fit an indirect utility function in consumer theory or a cost function in the theory of the firm usually involves the specification of a model in a parametric family. For the theory of the firm one can achieve this purpose specifying the cost function as $C(p, v) = f(\alpha_0 + \beta'w + (1/2)w'\Gamma w, v)$ where the function f is known, w is a vector of monotonically transformed prices p , and v is a function of output y that may depend on prices. The constant α_0 , the vector β , the matrix Γ , as well as any other unknown constants appearing in the function v , are parameters. A variant for consumer demand obtains dropping v and letting p be a vector of income normalized prices. Typical examples are the Translog cost (indirect utility) function and the generalized Box-Cox family. See Berndt and Khaled (1979), Gallant (1981), and Gallant (1982) for more details.

The objective of the econometric exercise of fitting an expenditure system is, usually, to estimate elasticities. One can completely specify these quantities with knowledge of the cost (indirect utility) function and its first and second order derivatives. From a statistical point of view the estimation of elasticities simplifies if the response model is taken from a parametric family and if the corresponding expenditure system is linear. This is the case of the theory of the firm with the use of the Translog.

A parametric family of cost (indirect utility) functions is said to be flexible if it endows the estimation procedure with nonparametric properties, that is, if we have a reasonably close approximation to target population quantities (strong consistency, for example) even if the true response function is not a member of the family for any parameter choice. Relatively recent studies indicate that none of the classical parametric families can achieve flexibility in this sense. See White (1980), Guilkey, Lovell and Sickles (1983), Gallant (1981,1982), and El Badawi, Gallant and Souza (1983). The reason for the lack of flexibility seems to be the failure of a Taylor series expansion to provide a uniform approximation simultaneously to a true response function and its first and second derivatives. In this context Gallant (1981) introduces the Fourier flexible form for indirect utility functions. The version for cost functions appeared in Gallant (1982). Essentially the Fourier flexible form superimposes a trigonomet-

ric polynomial to a Translog specification. The inclusion of sine and cosine terms endows the Translog family with the properties of Fourier series expansions. It is well known that a Fourier series can approximate a function in a Sobolev sense, that is, in a metric that allows a uniform approximation simultaneously for a function and its derivatives. See Edmunds and Moscatelli (1977). If the observational model used to approximate the factor (consumer) demand system has additive and independent errors one can prove strong consistency of estimators of elasticities, conditionally on a realization of prices and other covariates, for any of the standard statistical procedures. For this result to hold it is necessary to introduce a dependence of the order of the approximating Fourier series on the sample size. The question of asymptotic normality in a context where the number of parameters grow with sample size is more delicate. Only recently this problem has been properly addressed and solved under very special circumstances. These circumstances hypothesize factor demand systems and multivariate least squares (seemingly unrelated regressions) with a known variance matrix. Sufficient conditions for asymptotic normality require the growth of the number of parameters at a rate slower than any power of the sample size. See Gallant and Souza (1991) and also Andrews (1994) for a flavor of the regularity conditions that may be involved in general and for a strategy to prove consistency and asymptotic normality in semiparametric models.

It has been argued in the econometric literature that actual factor demand systems do not show additive errors. The problem of errors in variables is intrinsic to such systems. Any realistic application of seemingly unrelated techniques will face this condition which have not been contemplated yet in theoretical studies. In this context Chalfant and Gallant (1985) use a reoriented central composite design and Monte Carlo simulation to investigate the performance of the Fourier flexible cost function. They restrict their attention to the study of statistical biases. Two variables define the factor space: a measure of elasticity intensity σ and a technology λ chosen in the homothetic Box-Cox family. They fit a quadratic form in σ and λ to the response surface of biases generated by elasticities of substitution estimated via seemingly unrelated regressions. Here we extend Chalfant and Gallant's study. We investigate distributional results and deepen the discussion on statistical biases. Our objective, basing conclusions on Monte Carlo evidence, is to provide specific recommendations and warnings in regard to the practical use of the Fourier flexible cost function. We organize

our presentation as follows. In Section 2 we introduce the family of cost functions we use later. These are the general Box-Cox form and the Fourier flexible form. In Section 3 we describe in detail the experiment of Chalfant and Gallant (1985). In Section 4 we present our simulation results. We base our analysis on the absolute biases observed in estimates of substitution elasticities and in two measures of goodness of fit: the chi-square test statistic and the Shapiro-Wilk test statistic. Finally we summarize our findings in Section 5.

2 Cost Functions

Consider a production scheme involving n factor inputs. Let $p' = (p_1, \dots, p_n)$ be a vector of input prices and let $q' = (q_1, \dots, q_n)$ be the vector of input quantities. The total expenditure resulting from the choice of quantity vector q is $p'q$. Let $y = f(p_1, \dots, p_n)$ be the production function. The producer's problem, given a vector of factor input prices p and a level of output y , is to minimize total expenditure $p'q$ subject to $f(p_1, \dots, p_n) = y$. The minimum expenditure $C(p, y)$ corresponding to the solution $q(p, y)$ is the producer's cost function. The vector $q(p, y)$ defines the set of factor demand functions.

The theory of the firm may impose regularity conditions on $C(p, y)$. A minimal set of regularity conditions will impose linear homogeneity in prices, monotonicity in prices and output, and concavity in prices. A cost function $C(p, y)$ is said to be linear homogeneous if $C(\tau p, y) = \tau C(p, y)$ for any $\tau > 0$, monotonic if $\nabla_{p,y} C(p, y) > 0$, and concave if $\nabla_p^2 C(p, y)$ is negative semidefinite. Concavity and linear homogeneity imply that $\nabla_p^2 C(p, y)$ has rank at most $n-1$ with the price vector p being an eigenvector of root zero. A simplifying assumption on functional form imposes homotheticity (a general form of constant returns to scale), that is $C(p, y) = h(p)g(y)$ where $h(p)$ is linear homogeneous.

A linear homogeneous technology (cost function) will satisfy Shephard's lemma, that is $q(p, y) = \nabla_p C(p, y)$ or, equivalently, $S_i = \partial C(p, y) / \partial p_i / C(p, y)$, where S_i is the cost share of the i th input factor. Typically the econometricians that deal with expenditure systems will specify a parametric form for $C(p, y)$ and postulate an observational multivariate (possibly nonlinear) statistical model $s_i = \partial C(p, y) / \partial p_i / C(p, y) + \epsilon_i$, $i=1 \dots n-1$. In this latter expression s_i represents an observed share, p are observed prices, and one share equation has been

removed to avoid a singular model.

In some instances it is more convenient to express the cost function, prices, and output in log (base e) form. If $g(l, v)$ represents the log cost function, $l = \ln p$, and $v = \ln y$, the observational model becomes $s_i = \partial g(l, v) / \partial l_i + \epsilon_i$.

The most common statistical technique used in an attempt to determine $C(p, y)$ or $g(l, v)$ statistically is seemingly unrelated regressions. This estimation technique proceeds as follows. For observation $t=1 \dots n$ let $s'_t = (s_{1t}, \dots, s_{(n-1)t})$ be the vector of shares and $x'_t = (l'_t, v_t)$. We can write $s_t = \phi(x_t, \theta) + \epsilon_t$ where ϵ_t is an $n-1$ dimensional error vector and θ is a parameter derived from a completely specified the cost function. Let $S(\theta, V) = \sum_{t=1}^n (s_t - \phi(x_t, \theta))' V^{-1} (s_t - \phi(x_t, \theta))$ and let $\Psi(\theta) = 0$ be a set of restrictions on θ . The seemingly unrelated regressions estimate of θ obtains first finding $\hat{\theta}$ that minimizes $S(\theta, I)$ subject to $\Psi(\theta) = 0$, putting $\hat{V} = (1/T) \sum_{t=1}^n (s_t - \phi(x_t, \hat{\theta}))' (s_t - \phi(x_t, \hat{\theta}))$ and then minimizing $S(\theta, \hat{V})$ subject to $\Psi(\theta) = 0$. Iteration of this process, upon convergence, yields maximum likelihood estimates. See Gallant (1987).

As Gallant (1982) points out the main objective of the econometric attempt to fit share equations focus on the assessment of elasticities of substitution

$$\sigma_{ij} = \left(C(p, y) \frac{\partial^2 C(p, y)}{\partial p_i \partial p_j} \right) / \left(\frac{\partial C(p, y)}{\partial p_i} \frac{\partial C(p, y)}{\partial p_j} \right) \quad i, j = 1, \dots, n$$

and price elasticities $\eta_{ij} = S_j \sigma_{ij}$. Let $\Sigma = (\sigma_{ij})$. The assumption of concavity in prices for $C(p, y)$ is equivalent to Σ being negative semidefinite. Linear homogeneity implies that the share vector S must be an eigenvector of root zero.

2.1 The Generalized Box-Cox Family of Cost Functions

Berndt and Khaled (1979) present in detail the generalized Box-Cox parametric family of cost functions. Their formulation specifies

$$C(p, y) = (1 + \lambda G(p))^{1/\lambda} y^{\beta(y, p)}$$

where

$$G(p) = \alpha_0 + \sum_{i=1}^n \alpha_i p_i(\lambda) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} p_i(\lambda) p_j(\lambda)$$

$$\beta(y, p) = \beta + \frac{\theta}{2} \ln y + \sum_{i=1}^n \phi_i \ln p_i$$

$$p_i(\lambda) = \frac{2}{\lambda} (p_i^{\lambda/2} - 1)$$

We have $\lambda > 0$ and the matrix $\Gamma = (\gamma_{ij})$ is symmetric. Linear homogeneity in prices obtains if

$$\sum_{i=1}^n \alpha_i = 1 + \lambda \alpha_0 \quad \sum_{j=1}^n \gamma_{ij} = \frac{\lambda}{2} \alpha_i \quad \sum_{i=1}^n \phi_i = 0$$

When these restrictions are imposed,

$$C_\lambda(p, y) = \left(\frac{2}{\lambda} \sum_{i=1}^n \sum_{j=1}^n \gamma_{ij} p_i^{\lambda/2} p_j^{\lambda/2} \right)^{1/\lambda} y^{\beta(y, p)}$$

We obtain a homothetic technology imposing additionally $\phi_i = 0$ for all i .

Familiar technologies in the Box-Cox family are the Square Root Quadratic ($\lambda=2$) and the Generalized Leontief ($\lambda=1$). The Translog cost function can be obtained writing

$$G(p) = \frac{(C/y^\beta)^\lambda - 1}{\lambda}$$

and taking the limit as $\lambda \rightarrow 0$. The Translog cost function (in log form) is

$$g(l, v) = \alpha_0 + \alpha' l + \frac{1}{2} l' \Gamma l + \omega v + \frac{\theta}{2} v^2 + \sum_{i=1}^n l_i v$$

For the Translog cost function to be homogeneous in prices we must impose $\sum_i \alpha_i = 0$ and $\sum_j \gamma_{ij} = 0$.

The Monte Carlo experiment of Gallant and Chalfant (1985) deals only with homothetic technologies. For these cost functions substitution elasticities do not depend on output and in this context they take output to be unit. Expressions for shares and substitution elasticities in this case are ($\lambda > 0$)

$$S_i = \left(2 p_i^{\lambda/2} \sum_{j=1}^n \gamma_{ij} p_j^{\lambda/2} \right) / (\lambda C^\lambda(p, y)) \quad i = 1, \dots, n \quad (1)$$

$$\sigma_{ii} = 1 - \lambda + \gamma_{ii} \frac{p_i^\lambda}{S_i^2} + \frac{\lambda}{2 S_i} - \frac{1}{S_i} \quad i = 1, \dots, n \quad (2)$$

$$\sigma_{ij} = 1 - \lambda + \gamma_{ij} \frac{p_i^{\lambda/2} p_j^{\lambda/2}}{S_i S_j} - C_\lambda^{-\lambda}(p, y) \quad i \neq j \quad (3)$$

and for $\lambda = 0$

$$S_i = \alpha_i + \sum_{j=1}^n \gamma_{ij} l_j \quad i = 1, \dots, n \quad (4)$$

$$\sigma_{ii} = 1 + \frac{\gamma_{ii}}{S_i^2} - \frac{1}{S_i} \quad i = 1, \dots, n \quad (5)$$

$$\sigma_{ij} = 1 + \frac{\gamma_{ij}}{S_i S_j} \quad i \neq j \quad (6)$$

Using the normalization rule $C_\lambda(p) = 1$, given values for λ , σ_{ij} , p_i , and S_i , one can solve Equations (1)-(6) to find a unique set of γ_{ij} . Later we use this fact to derive population technologies. In this exercise the specification of the substitution matrix Σ will imply values for S_i . Notice that with knowledge of γ_{ij} we can use Equations (1) and (4) to generate share data.

2.2 The Fourier Flexible Form

Gallant (1982) introduces the Fourier flexible form for the analysis of factor demand systems as follows. As before let $x' = (l', v)$ where now $l_i = \ln p_i + \ln a_i$ and $v = \ln y + \ln a_{n+1}$. The constants a_1, \dots, a_{n+1} are location quantities chosen such that x becomes a vector with all coordinates strictly positive. One may take $\ln a_i = -\min_i \ln p_i + \zeta$ for any choice of $\zeta > 0$. The functional form of the Fourier flexible form with K parameters is

$$g_K(x, \theta) = u_0 + b'x + \frac{1}{2}x'Bx + \sum_{\alpha=1}^A \{u_{0\alpha} + \sum_{j=1}^J [u_{j\alpha} \cos(j\tau k'_\alpha x) - v_{j\alpha} \sin(j\tau k'_\alpha x)]\}$$

with

$$B = -\tau^2 \sum_{\alpha=1}^n u_{0\alpha} k_\alpha k'_\alpha$$

Here u_0 , $u_{0\alpha}$, b , B , $u_{j\alpha}$, and $v_{j\alpha}$ are parameters, k_α is a vector with integer components - a multi-index, and τ is a scale parameter chosen to make each x a point of $(0, 2\pi)^{n+1}$. The vector θ represents the collection of parameters written down in some order. In this paper we use $\zeta = 10^{-5}$ and $\tau = 6/\max_i x_i$.

A deep philosophical difference between the Fourier flexible form and other parametric cost functions lies in the dependence of K on the sample size T . We emphasize this dependence here writing $K = K_T$. The Fourier flexible form $g_K(x, \theta)$ approximates a true cost function in the Sobolev sense if $K_T \rightarrow \infty$ as $T \rightarrow \infty$. This result poses some problems in

applications. It is not clear in the literature of the subject how one should optimally choose K_T and therefore A , J , and the multi-indices k_α .

The Fourier flexible form will approximate a linear homogeneous cost function if we impose $\sum_i b_i = 1$ and keep only terms corresponding to multi-indices k_α for which $\sum_i k_{\alpha i} = 0$.

The Fourier flexible form expenditure system is linear since shares can be expressed as $S_i = \partial g_{K_T}(x, \theta) / \partial x_i = g'_i \theta$. Since also $\partial^2 g_{K_T}(x, \theta) / \partial x_i \partial x_j = h'_{ij} \theta$ we can compute elasticities of substitution as

$$\sigma_{ii}(\theta) = 1 + \frac{h'_{ii} \theta}{(g'_i \theta)^2} - \frac{1}{g'_i \theta} \quad \text{and} \quad \sigma_{ij}(\theta) = 1 + \frac{h'_{ij} \theta}{(g'_i \theta)(g'_j \theta)} \quad i \neq j.$$

It follows that if an estimator $\hat{\theta}$ is available, via seemingly unrelated regressions for example, we can estimate consistently the true elasticity of substitution σ_{ij} by $\sigma_{ij}(\hat{\theta})$ and try to approximate its variance by $\nabla' \sigma_{ij}(\hat{\theta}) \hat{\Omega} \nabla \sigma_{ij}(\hat{\theta})$ where $\hat{\Omega}$ is an estimate of the variance matrix of $\hat{\theta}$. One can compute price elasticities estimators in an analogous manner. See Gallant (1982) and El Badawi, Gallant and Souza (1983) for more details on this approach.

In our simulation study we use, as in Chalfant and Gallant (1985), two Fourier flexible forms: one for which $K_T = 13$ (FFF13) and another for which $K_T = 22$ (FFF22). In both cases the true technology is a function of a price vector of dimension three. For FFF22 we take $A = 6$ and $J = 1$. The choices of multi-indices are $k'_1 = (0 \ 1 \ -1)$, $k'_2 = (1 \ -1 \ 0)$, $k'_3 = (1 \ 0 \ -1)$, $k'_4 = (1 \ -2 \ 1)$, $k'_5 = (1 \ 1 \ -2)$, and $k'_6 = (2 \ -1 \ -1)$. The first three have norm two and the last three norm four. The norm of a multi-index is the sum of the absolute values of its components. The resulting (FFF22) form is

$$\begin{aligned} g_{22}(x, \theta) &= u_0 + b'x + \frac{1}{2}x' Bx \\ &+ u_{01} + 2\{u_1 \cos[\tau(x_2 - x_3)] - v_1 \sin[\tau(x_2 - x_3)]\} \\ &+ u_{02} + 2\{u_2 \cos[\tau(x_1 - x_2)] - v_2 \sin[\tau(x_1 - x_2)]\} \\ &+ u_{03} + 2\{u_3 \cos[\tau(x_1 - x_3)] - v_3 \sin[\tau(x_1 - x_3)]\} \\ &+ u_{04} + 2\{u_4 \cos[\tau(x_1 - 2x_2 + x_3)] - v_4 \sin[\tau(x_1 - 2x_2 + x_3)]\} \\ &+ u_{05} + 2\{u_5 \cos[\tau(x_1 + x_2 - 2x_3)] - v_5 \sin[\tau(x_1 + x_2 - 2x_3)]\} \\ &+ u_{06} + 2\{u_6 \cos[\tau(2x_1 - x_2 - x_3)] - v_6 \sin[\tau(2x_1 - x_2 - x_3)]\} \end{aligned}$$

The parametrization above for FFF22 leads to a singular design matrix. To avoid this

we reparametrize the elements of B and absorb the terms $u_{0\alpha}$ in the constant term. On B we impose symmetry and homogeneity.

The version FFF13 obtains from FFF22 reducing the order of the trigonometric polynomial. Here we consider only multi-indices with norm two. This is equivalent to set the coefficients $u_{04}, u_{05}, u_{06}, u_4, v_4, u_5, v_5, u_6,$ and v_6 to zero in the expression for FFF22. Also

$$B = -\tau^2 \begin{pmatrix} -u_{02} - u_{03} & u_{02} & u_{03} \\ u_{02} & -u_{01} - u_{02} & u_{01} \\ u_{03} & u_{01} & -u_{01} - u_{03} \end{pmatrix}$$

and the parametrization is identifiable.

3 Experimental Design

Basing their claims on the empirical evidence of past work Chalfant and Gallant (1985) argue that the presence of bias, and its relative size, in the statistical estimation of elasticities, may depend on the nature of the true technology, on the particular pattern of the elasticity matrix, and on the magnitude of each elasticity being estimated. In this context they design a response surface experiment depending on two quantitative factors: Technology - measured by variate λ , and Elasticity - measured by variate σ . The design space is the rectangle $[0, 2] \times [0, 2]$. Each choice of (σ, λ) defines a technology $C_\lambda(p_1, p_2, p_3)$ in the homothetic Box-Cox family for which the matrix of elasticities of substitution is $\Sigma(\sigma)$. They use a total of nine design points chosen as explained below. The idea is to fit the Fourier flexible form to data generated according to each of these "true" technologies and investigate the nature of the response surface defined by the absolute biases resulting from the estimation process. Of particular concern is the presence of a ridge of increasing bias in any particular direction (σ, λ) . They compute $\Sigma(\sigma)$ as follows. The own price elasticities are taken to be equal to $-\eta$ for all three prices. The diagonal elements of $\Sigma(\sigma)$ are then $\sigma_{ii} = -\eta/S_i$. The off diagonal elements have the same magnitude σ and Σ has one of the three patterns

$$I: \begin{pmatrix} \sigma_{11} & \sigma & \sigma \\ \sigma & \sigma_{22} & -\sigma \\ \sigma & -\sigma & \sigma_{33} \end{pmatrix} \quad II: \begin{pmatrix} \sigma_{11} & \sigma & -\sigma \\ \sigma & \sigma_{22} & \sigma \\ -\sigma & \sigma & \sigma_{33} \end{pmatrix} \quad III: \begin{pmatrix} \sigma_{11} & -\sigma & \sigma \\ -\sigma & \sigma_{22} & \sigma \\ \sigma & \sigma & \sigma_{33} \end{pmatrix}$$

The actual pattern used for each combination $(\sigma \lambda)$ was chosen randomly.

Given a value of $\sigma \neq 0$, Σ is completely determined by the placement of the negative sign. We illustrate this construction when

$$\Sigma = \begin{pmatrix} -\frac{\eta}{S_1} & \sigma & \sigma \\ \sigma & -\frac{\eta}{S_2} & -\sigma \\ \sigma & -\sigma & -\frac{\eta}{S_3} \end{pmatrix}$$

It follows (from $\sum S_i = 1$ and $\sum S = 0$) that

$$\begin{cases} -\eta + 0S_1 + \sigma S_2 + \sigma S_3 = 0 \\ -\eta + \sigma S_1 + \sigma S_2 + -\sigma S_3 = 0 \\ -\eta + \sigma S_1 + -\sigma S_2 + -0S_3 = 0 \\ -0\eta + S_1 + S_2 + S_3 = 1 \end{cases}$$

The solution for this system is $S_1 = 3/5$, $S_2 = S_3 = 1/5$ and $\eta = 2\sigma/5$. Thus for Pattern I

$$\Sigma(\sigma) = \frac{\sigma}{3} \begin{pmatrix} -2 & 3 & 3 \\ 3 & -6 & -3 \\ 3 & -3 & -6 \end{pmatrix}$$

For Patterns II and III we obtain, respectively,

$$\Sigma(\sigma) = \frac{\sigma}{3} \begin{pmatrix} -6 & 3 & -3 \\ 3 & -2 & 3 \\ -3 & 3 & -6 \end{pmatrix} \text{ and } \Sigma(\sigma) = \frac{\sigma}{3} \begin{pmatrix} -6 & -3 & 3 \\ -3 & -6 & 3 \\ -3 & 3 & -2 \end{pmatrix}$$

Notice that these three matrices are negative definite with rank two. At this point it is important to remark that when $\sigma = 0$ a random choice among Patterns I, II, and III was used to decide which share should have value $3/5$.

The design points in the factor space were chosen as follows. Consider first the central composite rotatable design described in Cochran and Cox (1957). This is a design commonly used in experimental statistics to approximate a response function by a quadratic form. The nine vectors $(0 - \sqrt{2})$, $(0 0)$, $(1 1)$, $(0 \sqrt{2})$, $(-1 1)$, $(-1 -1)$, $(1 -1)$, $(\sqrt{2} 0)$, and $(-\sqrt{2} 0)$ define the design. The transformation $(\frac{x+\sqrt{2}}{\sqrt{2}}, \frac{y+\sqrt{2}}{\sqrt{2}})$ maps the central composite into the square $[0, 2] \times [0, 2]$ producing the new design points (σ, λ) : $(1 0)$, $(1 1)$, $(1.70711 1.70711)$,

(1 2), (0.29289 1.70711), (0.29289 0), (.29289), (1.70711 0.29289), (2 1), and (0 1). This is the layout we will use. We refer to these points as design points 1-9 respectively. Known technologies in the layout are the Translog - Design 1, Leontief - Design 9, Square Root Quadratic - Design 4, and Generalized Leontief - Designs 2 and 8.

It is clear from Equations (1)-(6) of Section 2 and from the normalization rule $C_\lambda(p) = 1$ at $p = p_0$ that a design point (σ, λ) uniquely determines $C_\lambda(p)$ with the choice $\Sigma = \Sigma(\sigma)$.

Chalfant and Gallant (1985) consider two price series to generate data in the Monte Carlo process. The idea is to mimic the behavior of expected and actual prices so that share data is simulated with expected prices and fit to actual prices. A typical case of errors in variables and non additive regression errors. The series of actual prices p_t is generated according to the model $\ln p_t = \beta + u_t$, $u_t = Ru_{t-1} + \epsilon_t$, where $\beta' = (0.09760778 \ -0.00721513 \ 0.37572201)$,

$$R = \begin{pmatrix} 0.56627 & 0.055096 & -0.017574 \\ 0.13356 & 0.502859 & 0.129266 \\ -0.14791 & 0.110825 & 0.934661 \end{pmatrix}$$

and ϵ_t is a three dimensional multivariate normal random vector with mean zero and variance $V = F'F$. Here

$$F = \begin{pmatrix} -0.00135824 & -0.08386093 & -0.00054760 \\ 0.00067686 & 0.00017299 & -0.02817099 \\ -0.01265727 & 0.00206979 & -0.00030284 \end{pmatrix}$$

For p_0 we take $\exp(\beta)$. This is the price vector used to obtain the population technologies in Table 1. The (vector) stochastic process above was adapted by Chalfant and Gallant (1985) from a real price series studied in Berndt and Wood (1975).

Following Huber (1981) $K_T \approx T^{2/3}$. In this context a basic sample of 25 observed prices is generated to fit FFF13 and extended to 48 to fit FFF22. With only two shares retained the number of effective observations in each case becomes 50 and 96 respectively.

The series of expected prices \hat{p}_t is a function of observed prices p_t . We take $\ln(\hat{p}_t) = \ln(p_t) + \zeta_t$ where $\zeta_t = \beta * y_t$. Here y_t is a multivariate normal random vector with mean zero and variance matrix αI . Shares are generated using the formula $s_{it} = \partial C_\lambda(\hat{p}_t) / \partial p_i$ which is Shephard's lemma. Chalfant and Gallant (1985) generate 5000 replications and set α to

0.1. This choice of α , in their case, allows estimation of elasticities with accuracy of three digits when the true technology is the Translog. Here we take 1000 Monte Carlo replications. We achieve the same accuracy for the Translog case with $\alpha = 0.01$ for both fits (FFF13 and FFF22). The number of replications we choose is in accordance with the bootstrap literature, see Efron (1990) and Efron and Tibshirani (1993), and keeps the simulation process at a PC manageable size.

4 Simulation Results

Our simulation study mimics the basic structure (model) of Chalfant and Gallant (1985). We use SAS - ETS to compute seemingly unrelated regressions, SAS - PROC IML to compute standard errors of elasticities of substitution and nonparametric density estimates, and SAS - STAT to evaluate empirical distributions. In Subsection 4.1 we present our findings in regard to biases in estimation. Our discussion has more detail than that of Chalfant and Gallant (1985). In Subsection 4.2 we show our results regarding other distributional aspects.

4.1 Biases in the Estimation of Elasticities

The variable of concern here is the absolute bias $|\sigma_{ij} - \bar{\sigma}_{ij}|$ where σ_{ij} is the true elasticity of substitution and $\bar{\sigma}_{ij}$ is an average of 1000 seemingly unrelated estimates $\hat{\sigma}_{ij}$.

General: Table 2 shows the evolution of two summary statistics computed from the basic results reported in Félix Souza (1993). Those are the average (over σ'_{ij} s) bias per technology (λ), identified with the design, and the average bias relative to the Translog which functions as a base case or control. The overall impression is that FFF13 has a better performance than FFF22. The worst case for FFF13 is technology five which on average shows a bias close to 0.03. This is about eight times the bias we measure when we use FFF13 and the data is generated according to the Translog. For FFF22 two design points seem to be outliers. Technologies four and eight. The form FFF22 produces three bad estimates of elasticities in each of these cases leading to average biases of 0.08 and 0.14 respectively. These figures are approximately 45 and 77 times the average bias we measure when we fit FFF22 to data generated from the Translog. Chalfant (1983) also reports an increase in bias

when using the FFF22 but our results are more dramatic. He attributes the misfit to errors in variables. In our case other sources of instability may also be confounded in the process. Potential candidates are specification bias and multicollinearity.

Analysis of Variance: The Analysis of Variance results are shown in Tables 3, 4, and 5. The statistical findings confirm the visual impression from Table 2. In Table 3 we notice a highly significant interaction between Models with two levels - FFF13 and FFF22 and Technology.

Table 4 shows our findings for FFF13. The overall F test does not indicate significant differences among technologies although Dunnett's procedure (not shown) indicate a real difference (5% level) between technology five and the Translog (control).

In Table 5 we have ANOVA for FFF22. The overall F test is significant and Dunnett's procedure (not shown) points to technologies four and eight as differing significantly (5% level) from the control.

Response Surface Analysis: Tables 6 and 7 show fits of a quadratic surface in λ and σ for FFF13 and FFF22 respectively. Model FFF13 passes the lack of fit test and the stationary point is a saddle point outside the region under study. The overall results for FFF13 agree with Chalfant and Gallant (1985). Model FFF22 does not pass the lack of fit test. A ridge of maximum response is observed in the direction of $(\sigma = 1.956 \ \lambda = 1.292)$. Chalfant and Gallant (1985) find different results in this instance.

4.2 Distributional Aspects

To study deviations from population probabilities we compute two measures of goodness of fit. A measure we call X^2 which is the chi-square test statistic and the Shapiro-Wilk test statistic W . For model FFF13 the variable X^2 measures the overall discrepancy from the fit of a t distribution with 39 df to observed ratios of the form $t_{ij} = (\hat{\sigma}_{ij} - \sigma_{ij})/s(\hat{\sigma}_{ij})$ where $s(\hat{\sigma}_{ij})$ is the seemingly unrelated regressions estimate of the standard error of $\hat{\sigma}_{ij}$. For FFF22 variable X^2 measures a similar quantity for the t distribution with 79 df. We consider in both cases 14 nonoverlapping classes. Of course these classes would be about the same if we had used the standard normal distribution. The Shapiro-Wilk test statistic W measures departure from normality when we consider only the variate $\hat{\sigma}_{ij}$.

4.2.1 Variable X^2

General: The empirical probabilities associated with t_{ij} in many cases differ markedly from tables of the t distribution. This fact is illustrated in Exhibit 1 where density estimates are reported for Design 8. With a few exceptions the pattern is typical of other designs.

Table 8 shows the evolution of averages of $\ln X^2$ per technology or design point. The path for FFF22 dominates indicating a better fit for FFF13. Only three design points (5, 6, and 9) show averages below common significance levels. The Translog is a misfit for both functional forms.

Analysis of Variance: As in the case for the absolute bias the analysis of variance for $\ln X^2$, in Table 9, seem to confirm the visual impression. The interaction Model \times Technology is not significant and in this context we performed only a combined analysis. Dunnett's procedure (not shown) indicates real differences from the Translog only for design points five, six, and nine which are the technologies with acceptable levels for X^2 .

Response Surface Analysis: Tables 10 and 11 show fits of a quadratic surface in λ and σ to $\ln X^2$. FFF13 passes the lack of fit test and the stationary point ($\sigma = 1.288618$ $\lambda = 0.945635$) is a maximum. This is closest to the Generalized Leontief. FFF22 does not pass the lack of fit test and a ridge of maximum response is found in the direction of ($\sigma = 1.34$ $\lambda = 1.94$) which defines a technology close to the Square Root Quadratic.

4.2.2 Variable W

General: The quantities $\hat{\sigma}_{ij}$ show density estimates much closer to normality than the t_{ij} . Exhibit 2 for Design 8 is typical.

Table 12 shows the evolution of averages of W per technology. The path for FFF22 dominates the path for FFF13. For FFF22 only design points one and two show significant average levels. Design point eight is an outlier for FFF13. The overall impression is that the bias in the estimation of both σ_{ij} and the standard error of $s(\hat{\sigma}_{ij})$, rather than the polynomial rate of dependence to the sample size, may be the cause for worse results with X^2 .

Analysis of Variance: The analysis of variance for variable W is reported in Table 13. The interaction Model \times Technology is not significant. Dunnett's procedure (not shown) indicate that the only difference not real is between design points one and two. Again the control

treatment is the Translog technology.

Response Surface Analysis: The fit is reported in Tables 14 and 15. Model FFF13 does not pass the lack of fit test. A ridge of minimum response is found in the direction of design point two. Model FFF22 passes the lack of fit test and its stationary point is a minimum and has coordinates $\sigma = 0.9812826$ and $\lambda = 0.871032$. As with X^2 we see that the direction of worst results is close to the Generalized Leontief.

5 Conclusions

The Fourier flexible form seems to perform very well for $K_T = 13$, particularly in regard to point estimation of elasticities. The inclusion of additional trigonometric terms seems to lead to unstable estimates. The classical Wald test statistics based on seemingly unrelated estimates do not follow a t-distribution. The better distributional results obtained with the Shapiro-Wilk test statistic, which properly centers and scales the estimates $\hat{\sigma}_{ij}$, is an indication that the use of bootstrap techniques to correct for bias and to set confidence intervals will provide a more reliable statistical inference than the classical delta method. In general our analysis indicate that distributional results are particularly bad in the direction of Generalized Leontief cost functions. The discrepancies observed are too large to be attributed to errors in variable alone.

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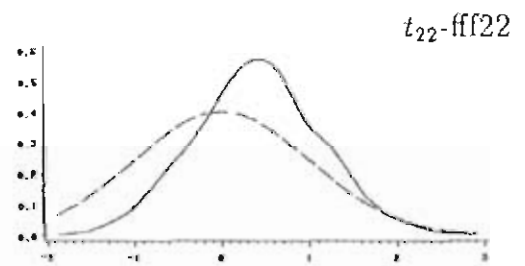
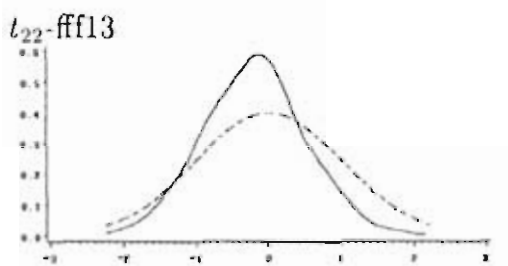
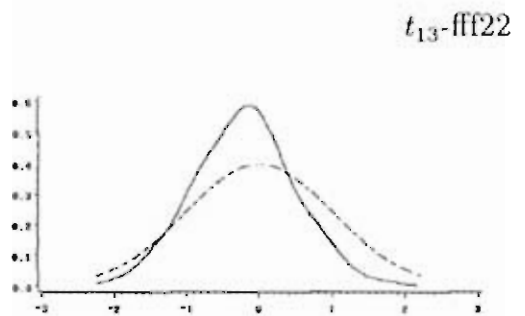
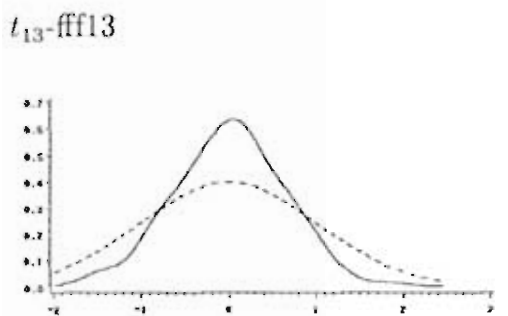
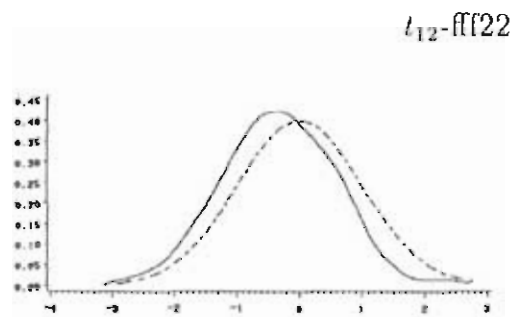
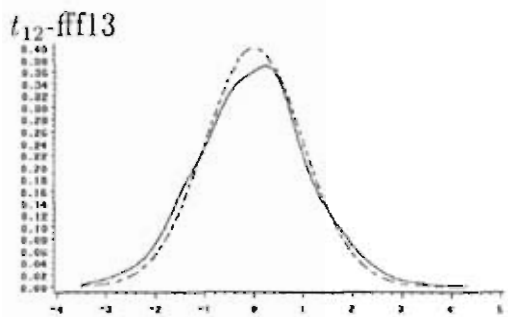
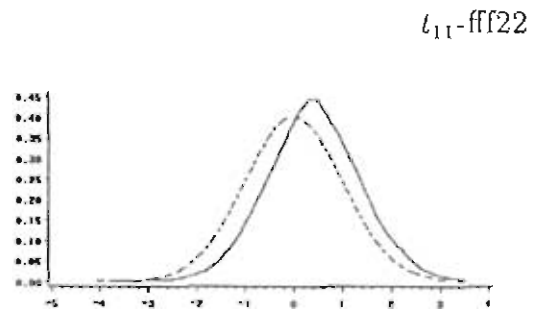
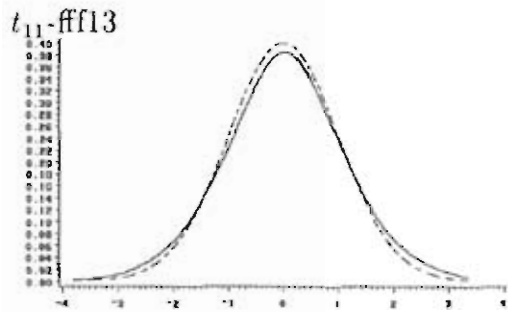
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Exhibit 1: Nonparametric Density Estimates for the distribution of t_{ij} for Design 8.
 Normal Kernel. The Dashed Curve is the Standard Normal.



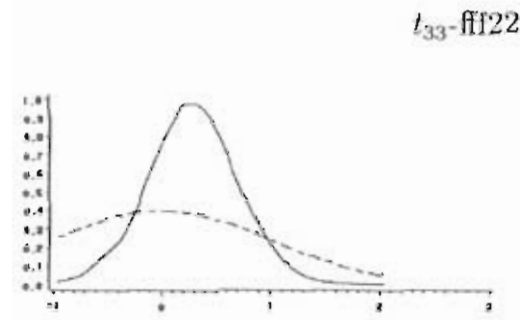
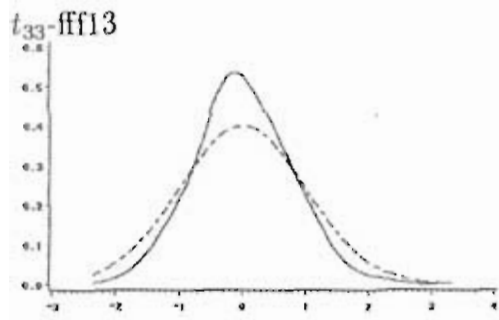
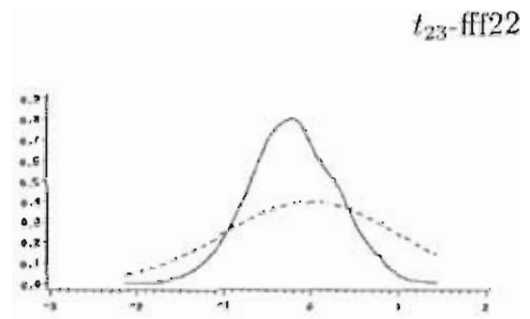
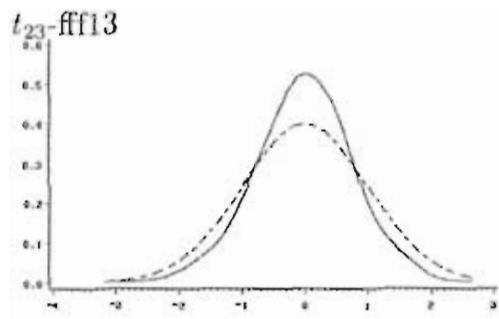
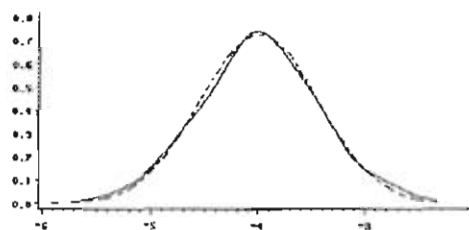
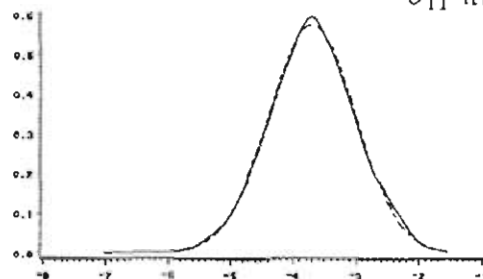


Exhibit 2: Nonparametric Density Estimates for the distribution of $\hat{\sigma}_{ij}$ for Design 8. Normal Kernel. The Dashed Curve is the Normal Density with Mean and Variance Computed from the Data.

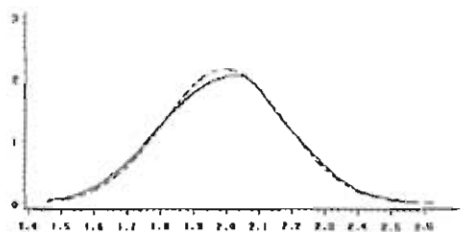
$\hat{\sigma}_{11}$ -fff13



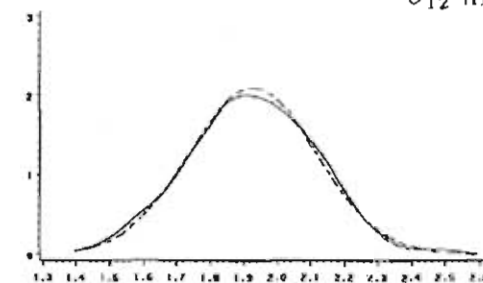
$\hat{\sigma}_{11}$ -fff22



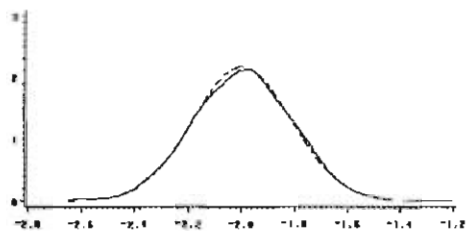
$\hat{\sigma}_{12}$ -fff13



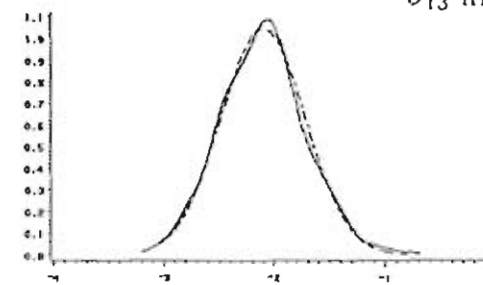
$\hat{\sigma}_{12}$ -fff22



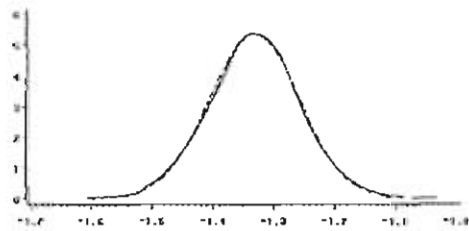
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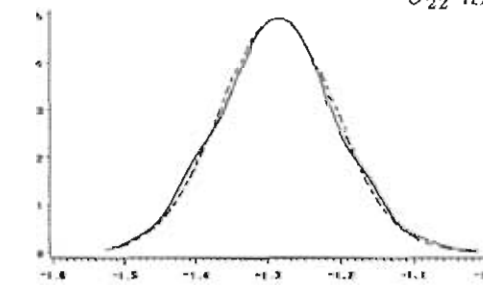
$\hat{\sigma}_{13}$ -fff22



$\hat{\sigma}_{22}$ -fff13



$\hat{\sigma}_{22}$ -fff22



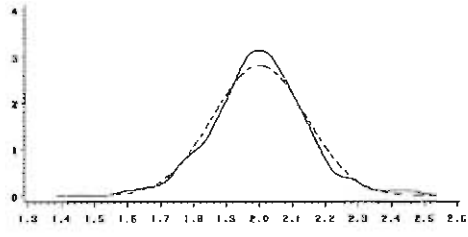
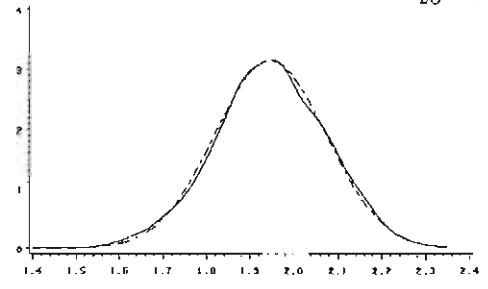
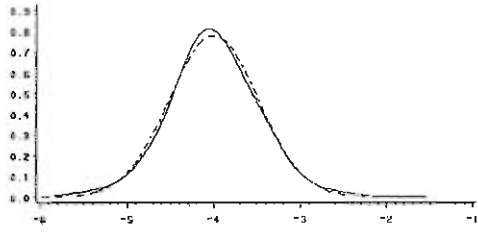
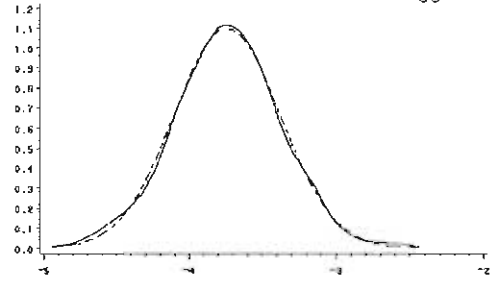
$\hat{\sigma}_{23}\text{-ff13}$  $\hat{\sigma}_{23}\text{-ff22}$  $\hat{\sigma}_{33}\text{-ff13}$  $\hat{\sigma}_{33}\text{-ff22}$ 

Table 1. Population Technologies. For the Translog $\alpha_0 = 0.756928$, $\alpha_1 = 0.191614$,
 $\alpha_2 = 0.208386$, and $\alpha_3 = 0.6$.

Design	Pattern	γ_{11}	γ_{12}	γ_{13}	γ_{22}	γ_{23}	γ_{33}
1	III	0.0800	-0.0800	0	0.0800	0	0
2	I	0.1088	0.2294	0.1894	0.04003	-0.0665	0.0275
3	I	0.0667	0.3142	0.2266	-0.0937	-0.0342	-0.0487
4	III	-0.0329	0	0.1495	-0.0406	0.1660	0.0566
5	I	0.2699	0.1302	0.0939	0.0405	0.0142	0.0211
6	III	0.7896	-0.2696	-0.3167	0.8143	-0.3216	1.1456
7	I	-1.0095	0.8086	0.7645	0.0401	-0.6248	0.0358
8	II	-0.1088	0.4588	-0.1263	-0.3626	0.3992	-0.0824
9	II	0.1814	0	0	0.6043	0	0.1374

Table 2. Average Absolute and Relative Biases for the Fourier Flexible Form.

Design	Absolute	Absolute	Relative	Relative
	Bias	Bias	Bias	Bias
	FFF13	FFF22	FFF13	FFF22
1	0.00369950	0.0018189	1.0000	1.0000
2	0.01527360	0.0232128	4.1286	12.7620
3	0.00083742	0.0299361	2.2636	16.4584
4	0.00730160	0.0820313	1.9737	45.0994
5	0.02919300	0.0181192	7.8911	9.9616
6	0.00802730	0.0098951	2.1698	5.4402
7	0.00787500	0.0038115	2.1287	2.0955
8	0.01153290	0.1398890	3.1174	76.9086
9	0.00936490	0.0304001	2.5314	16.7135

Table 3: Analysis of Variance: Interaction Effects for Absolute Bias.

Source	df	Sum of Squares	Mean Square	F Ratio	Pr > F
Model	1	0.01895627	0.01895627	14.32	0.0003
Technology	8	0.05003558	0.00625445	4.73	0.0001
Model×Technology	8	0.05057292	0.00632161	4.78	0.0001
Error	90	0.11909705	0.00132330		
Corrected Total	107	0.23866182			

Table 4: Analysis of Variance for FFF13 - Response is the Absolute Bias.

Source	df	Sum of Squares	Mean Square	F Ratio	Pr > F
Technology	8	0.00266628	0.00033329	1.56	0.1651
Error	45	0.00963404	0.00021409		
Corrected Total	53	0.01230033			

Table 5: Analysis of Variance for FFF22 - Response is the Absolute Bias.

Source	df	Sum of Squares	Mean Square	F Ratio	Pr > F
Technology	8	0.09794222	0.01224278	5.03	0.0002
Error	45	0.10946300	0.00243251		
Corrected Total	53	0.20740522			

Table 6: Response Surface Analysis - FFF13. Numbers in (.) are standard errors. Model is

$$| \text{Bias} | = a_0 + a_1\lambda + a_2\sigma + a_3\lambda^2 + a_4\lambda\sigma + a_5\sigma^2. R^2 = 0.1364.$$

Source	df	Sum of Squares	Mean Square	F Ratio	Pr > F
Lack of Fit	3	0.000988	0.000329	1.539	0.2175
Pure Error	45	0.009634	0.000214		
Total Error	48	0.010622	0.000221		
a_0	a_1	a_2	a_3	a_4	a_5
-0.005830	0.029217	0.011425	-0.007077	-0.010333	-0.002128
(0.010519)	(0.015778)	(0.015778)	(0.007121)	(0.006073)	(0.007121)

Table 7: Response Surface Analysis - FFF22. Numbers in (.) are standard errors. Model is

$$| \text{Bias} | = a_0 + a_1\lambda + a_2\sigma + a_3\lambda^2 + a_4\lambda\sigma + a_5\sigma^2. R^2 = 0.2354.$$

Source	df	Sum of Squares	Mean Square	F Ratio	Pr > F
Lack of Fit	3	0.049123	0.016374	6.731	0.0008
Pure Error	45	0.109463	0.002433		
Total Error	48	0.158586	0.003304		
a_0	a_1	a_2	a_3	a_4	a_5
0.010201	0.027847	-0.056332	-0.005336	0.008950	0.037884
(0.040644)	(0.060966)	(0.060966)	(0.027516)	(0.023466)	(0.027516)

Table 8: Averages for $\ln X^2$.

Design	FFF13	FFF22
1	4.6946624	5.1636203
2	4.8546326	5.1677325
3	3.9233675	4.2385643
4	4.1387512	6.3972947
5	2.9973857	2.9873962
6	2.6377892	2.6205318
7	3.8100154	3.6118297
8	4.7126976	5.6266261
9	3.0075468	3.1578674

Table 9: Analysis of Variance: Interaction Effects for $\ln X^2$.

Source	df	Sum of Squares	Mean Square	F Ratio	Pr > F
Model	1	5.864931	5.864931	4.12	0.0452
Technology	8	103.234847	12.904356	9.07	0.0001
Model \times Technology	8	13.382641	1.672830	1.18	0.3223
Error	90	128.010129	1.422335		
Corrected Total	107	250.492547			

Table 10: Response Surface Analysis - FFF13. Numbers in (.) are standard errors. Model is $\ln X^2 = a_0 + a_1\lambda + a_2\sigma + a_3\lambda^2 + a_4\lambda\sigma + a_5\sigma^2$. $R^2 = 0.2163$.

Source	df	Sum of Squares	Mean Square	F Ratio	Pr > F
Lack of Fit	3	8.871097	2.957032	1.712	0.1781
Pure Error	45	77.738425	1.727521		
Total Error	48	86.609522	1.804365		

a_0	a_1	a_2	a_3	a_4	a_5
1.760963	1.739899	3.705639	-0.836075	-0.123121	-1.392659
(0.949826)	(1.424743)	(1.424743)	(0.643038)	(0.548381)	(0.643038)

Table 11: Response Surface Analysis - FFF22. Numbers in (.) are standard errors. Model

$$\text{is } \ln X^2 = a_0 + a_1\lambda + a_2\sigma + a_3\lambda^2 + a_4\lambda\sigma + a_5\sigma^2. R^2 = 0.3481.$$

Source	df	Sum of Squares	Mean Square	F Ratio	Pr > F
Lack of Fit	3	37.166524	12.388841	11.090	0.0000
Pure Error	45	50.271704	1.117149		
Total Error	48	87.438228	1.821630		

a_0	a_1	a_2	a_3	a_4	a_5
1.915438	0.850512	4.156474	-.248192	0.129934	-1.636403
(0.954359)	(1.431543)	(1.431543)	(0.646107)	(0.550999)	(0.646107)

Table 12: Averages for W.

Design	FFF13	FFF22
1	0.9414173	0.9690227
2	0.9460717	0.9648798
3	0.9872592	0.9877892
4	0.9838588	0.9852052
5	0.9868368	0.9851093
6	0.9866263	0.9859878
7	0.9875977	0.9884658
8	0.9692697	0.9870367
9	0.9849143	0.9870627

Table 13: Analysis of Variance: Interaction Effects for W.

Source	df	Sum of Squares	Mean Square	F Ratio	Pr > F
Model	1	0.00148329	0.00148329	4.30	0.0411
Technology	8	0.01744427	0.00218053	6.31	0.0001
Model×Technology	8	0.00284368	0.00035546	1.03	0.4201
Error	90	0.03107763	0.00034531		
Corrected Total	107	0.05284887			

Table 14: Response Surface Analysis - FFF13. Numbers in (.) are standard errors. Model

$$\text{is } W = a_0 + a_1\lambda + a_2\sigma + a_3\lambda^2 + a_4\lambda\sigma + a_5\sigma^2, R^2 = 0.2368.$$

Source	df	Sum of Squares	Mean Square	F Ratio	Pr > F
Lack of Fit	3	0.006684	0.002228	3.993	0.0132
Pure Error	45	0.025108	0.000558		
Total Error	48	0.031793	0.000662		

a_0	a_1	a_2	a_3	a_4	a_5
1.003676	-0.039486	-0.082646	0.025174	-0.000274	0.039628
(0.018198)	(0.027297)	(0.027297)	(0.012320)	(0.010507)	(0.012320)